CPE 310: Numerical Analysis for Engineers

Chapter 2: Solving Sets of Equations

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“simultaneous linear equations”

An extremely important topic is *how to solve a large system of equations*

Linear systems are perhaps the most widely applied numerical procedures when real-world situations are to be simulated.

A matrix is a **rectangular array of numbers** in which not only the **value** of the number is important but also its **position** in the array

CAPITAL LETTERS are used to refer to matrices

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3m} \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{4m} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nm} \end{bmatrix} = [a_{ij}]$$

$i = 1,2,\dots,n$ $j = 1,2,\dots,m$

a_{ij}

$i - th$ Row

$j - th$ Column

Size of the matrix is described by **the number of its rows and columns**

A matrix of *n* rows and *m* columns is said to be of size: *n* x *m*

Two matrices of the same size may be **added** or **subtracted**

$A = [a_{ij}]$ and $B = [b_{ij}]$

$$C = A + B = [a_{ij} + b_{ij}] = [c_{ij}]$$

$$D = A - B = [a_{ij} - b_{ij}] = [d_{ij}]$$

$$A = \begin{bmatrix} 4 & 7 & -5 \\ -4 & 2 & 12 \end{bmatrix}$$

A is 2×3

$$B = \begin{bmatrix} 1 & 5 & 4 \\ 2 & -6 & 3 \end{bmatrix}$$

B is 2×3

$$C = A + B = \begin{bmatrix} 5 & 12 & -1 \\ -2 & -4 & 15 \end{bmatrix}$$

$$D = A - B = \begin{bmatrix} 3 & 2 & -9 \\ -6 & 8 & 9 \end{bmatrix}$$

$$A = [a_{ij}] \text{ and } B = [b_{ij}]$$

A is $n \times m$ $\quad$ B is $m \times r$

$$C = [a_{ij}][b_{ij}] = [c_{ij}] = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1}) & \cdots & (a_{11}b_{1r} + \cdots + a_{1m}b_{mr}) \\ (a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2m}b_{m1}) & \cdots & (a_{21}b_{1r} + \cdots + a_{2m}b_{mr}) \\ \vdots & \vdots & \vdots \\ (a_{n1}b_{11} + a_{n2}b_{21} + \cdots + a_{nm}b_{m1}) & \cdots & (a_{n1}b_{1r} + \cdots + a_{nm}b_{mr}) \end{bmatrix}$$

$$\boxed{C \text{ is } n \times r}$$

$$\boxed{c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}, i = 1,2,\cdots n, j = 1,2,\cdots r}$$

$$AB \neq BA$$

If a matrix is multiplied by a scalar (a pure number), the product is a matrix, each element of which is the scalar times the original element

If $kA = C, c_{ij} = ka_{ij}$

Column Vector

A matrix with only one column, $n \times 1$ in size

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$A \text{ is } 4 \times 1$

When the unqualified term vector is used, it nearly always means a ***column vector***

Row Vector

A matrix with only one row, $1 \times n$ in size

$$A = \begin{bmatrix} 1 & 2 & 5 & 6 \end{bmatrix}$$

$A \; is \; 1 \times 4$

$$A = \begin{bmatrix} 3 & 7 & 1 \\ -2 & 1 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -2 \\ 0 & 3 \\ 1 & -1 \end{bmatrix} \qquad x = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$A * B = \begin{bmatrix} 16 & 14 \\ -13 & 10 \end{bmatrix}$$

$$B * A = \begin{bmatrix} 19 & 33 & 11 \\ -6 & 3 & -9 \\ 5 & 6 & 4 \end{bmatrix}$$

$$A * x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$A * y = \begin{bmatrix} 3y_1 + 7y_2 + y_3 \\ -2y_1 + y_2 - 3y_3 \end{bmatrix}$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$Ax = b$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 0 \\ -1 & 3 & 2 \end{bmatrix} * x = \begin{bmatrix} 14 \\ -7 \\ 2 \end{bmatrix}$$

is the same as the set of equations

$$3x_1 + 2x_2 + 4x_3 = 14$$
$$x_1 - 2x_2 = -7$$
$$-x_1 + 3x_2 + 2x_3 = 2$$

A very important special case is the multiplication of two vectors. The first must be a row vector if the second is a column vector, and each must have the same number of components. It gives a "matrix" of one row and one column.

$$\begin{bmatrix} 1 & 3 & -2 \end{bmatrix} * \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}$$

Inner Product

"*Scalar Product of Vectors*"

A very important special case is the multiplication of two vectors. The first must be a row vector if the second is a column vector, and each must have the same number of components.

If we reverse the order of multiplication of these two vectors, we call result *outer product*

$$\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} * \begin{bmatrix} 1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 12 & -8 \\ -1 & -3 & 2 \\ 3 & 9 & -6 \end{bmatrix}$$

Outer Product

Diagonal Matrix

A **Square Matrix** with only the *diagonal terms "elements"* are nonzero

The **diagonal elements** are the line of elements a_{ii} from upper left to lower right of the matrix

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Identity Matrix

A **Square Matrix** with the diagonal elements are each equal to **unity** while all off-diagonal elements are **zero**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

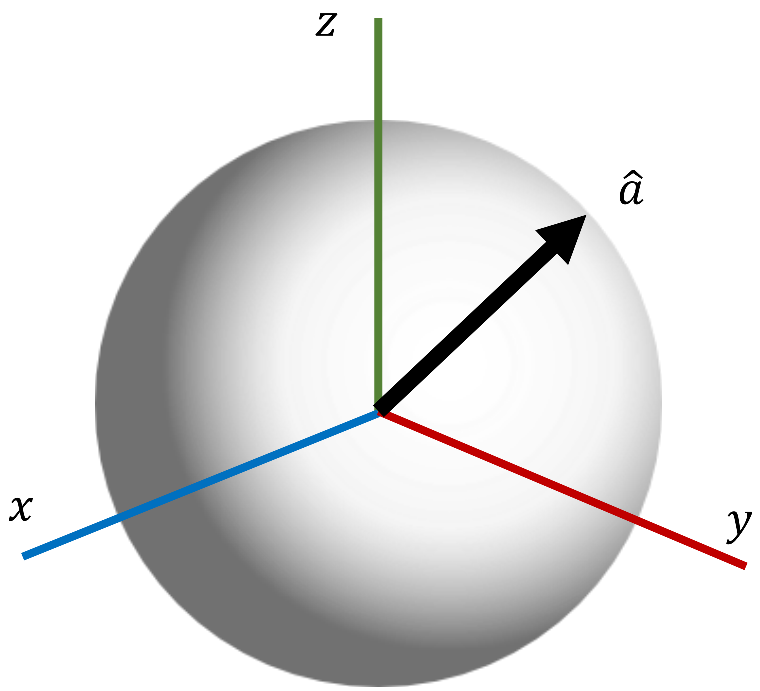
Identity Matrix of order 3

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

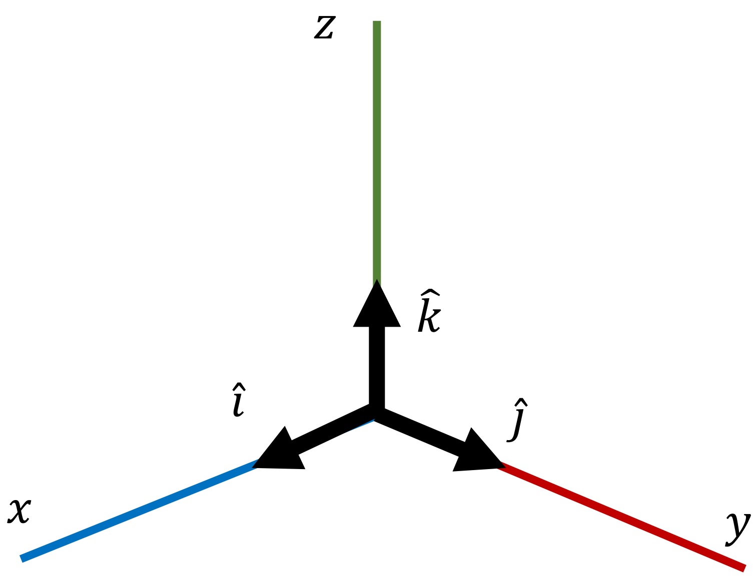
Identity Matrix of order 4

Unit Vector

A vector whose length is one



$$\sqrt{a^2 + b^2 + c^2} = 1$$



Unit Basis Vector

A vector that has all its **elements equal to zero except one element**, which has a value of **unity**

$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

What are the distinct **unit basis vectors** for order-4 vectors?

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Lower-Triangular Matrix

If all the elements above the diagonal are zero

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 6 & 0 \\ -2 & 1 & -4 \end{bmatrix}$$

Upper-Triangular Matrix

If all the elements below the diagonal are zero

$$U = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Tridiagonal Matrix

A **square matrix** that have nonzero elements only on the diagonal and in the positions adjacent to the diagonal

$$\begin{bmatrix} -4 & 2 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 2 & -4 \end{bmatrix}$$

Transpose of a Matrix

The matrix that results when the rows are written as columns (or, alternatively, when the columns are written as rows)

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & -3 \\ 1 & 1 & 2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 1 \\ 4 & -3 & 2 \end{bmatrix}$$

The transpose of A^T is just A itself

Trace of a Square Matrix

The sum of the elements on its main diagonal

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & -3 \\ 1 & 1 & 2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 1 \\ 4 & -3 & 2 \end{bmatrix}$$

$$tr(A) = 3 + 2 + 2 = 7 \qquad tr(A^T) = 3 + 2 + 2 = 7$$

*The **trace remains the same** if a square matrix is **transposed***

Examples

$$3 * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 2 \\ 4 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 4 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -4 & -5 \\ 7 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 2 & 6 \end{bmatrix} * \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} * \begin{bmatrix} 0 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 1 & 5 \end{bmatrix}$$

Examples

$$\begin{bmatrix} 0 & 3 \\ -1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 \\ 3 & -2 \end{bmatrix} * \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \text{ is } \textbf{NOT} \text{ defined}$$

Division of a matrix by another matrix is not defined, but we will discuss the **inverse of a matrix** later

Determinant

The *determinant* of a square matrix is a number

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A = \begin{bmatrix} -5 & 3 \\ 7 & 2 \end{bmatrix}, \det(A) = \begin{vmatrix} -5 & 3 \\ 7 & 2 \end{vmatrix} = (-5)(2) - (7)(3) = -31$$

Determinant

The *determinant* of a square matrix is a number

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + \mathrm{c}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei - afh - bdi + bfg + cdh - ceg$$

Determinant

The *determinant* of a square matrix is a number

$$\det(A) = \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & k \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Example

$$A = \begin{bmatrix} 3 & 0 & -1 & 2 \\ 4 & 1 & 3 & -2 \\ 0 & 2 & -1 & 3 \\ 1 & 0 & 1 & 4 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 3 & 0 & -1 & 2 \\ 4 & 1 & 3 & -2 \\ 0 & 2 & -1 & 3 \\ 1 & 0 & 1 & 4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 3 \\ 0 & 1 & 4 \end{vmatrix} - 0\begin{vmatrix} 4 & 3 & -2 \\ 0 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix} + (-1)\begin{vmatrix} 4 & 1 & -2 \\ 0 & 2 & 3 \\ 1 & 0 & 4 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 & 3 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 3\left\{1\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} + (-2)\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}\right\} + (-1)\left\{4\begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} + (-2)\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}\right\}$$
$$-2\left\{4\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}\right\}$$

$$= 3\{(1)(-7) - (3)(8) + (-2)(2)\} + (-1)\{(4)(8) - (1)(-3) + (-2)(-2)\}$$
$$-2\{(4)(2) - (1)(1) + (3)(-2)\}$$

$$= 3(-7 - 24 - 4) + (-1)(39) - 2(1) = -146$$

Determinant

The *determinant* of a square matrix is a number

$$\det(A) = \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & k \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Positive Sign if (**Row Number + Column Number***)* is **EVEN**

Negative Sign if (**Row Number + Column Number**) is **ODD**

Determinant Property

For square matrices A and B or equal size

$$\det(AB) = \det(A) * \det(B)$$

$$A = \begin{bmatrix} -5 & 3 \\ 7 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 5 \\ 2 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 11 & -7 \\ -3 & 47 \end{bmatrix}$$

$$\det(AB) = 496$$

$$\det(AB) = \begin{vmatrix} -5 & 3 \\ 7 & 2 \end{vmatrix} * \begin{vmatrix} -1 & 5 \\ 2 & 6 \end{vmatrix}$$

$$\det(AB) = (-31) * (-16) = 496$$

Determinant Property

If A is a **triangular matrix of order *n*** such that $a_{ij} = 0$ whenever $i > j$ or $i < j$

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

Lower-Triangular Matrix

If all the elements above the diagonal are zero

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 6 & 0 \\ -2 & 1 & -4 \end{bmatrix}$$

$$\det(L) = (1)(6)(-4) = -24$$

Upper-Triangular Matrix

If all the elements below the diagonal are zero

$$U = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(L) = (1)(-1)(1) = -1$$

Characteristic Polynomial

The characteristic polynomial is always of degree n if A is $n\times n$

$$p_A(\lambda) = |A - \lambda \boldsymbol{I}| = \det(A - \lambda \boldsymbol{I})$$

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$$

$$p_A(\lambda) = |A - \lambda \boldsymbol{I}| = \det(\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$$

$$= \det(\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix})$$

$$= \det(\begin{bmatrix} 1 - \lambda & 3 \\ 4 & 5 - \lambda \end{bmatrix})$$

$$= (1 - \lambda)(5 - \lambda) - (3)(4) = \lambda^2 - 6\lambda - 7$$

If we set the **characteristic polynomial** to zero and solve for the roots, we get the **eigenvalues of A**

The eigenvalues are most important in applied mathematics

$$p_A(\lambda) = \lambda^2 - 6\lambda - 7 = 0$$

$$\boxed{\Lambda(A) = \{7, -1\}}$$

The sum of eigenvalues for any matrix is equal to its trace

$$tr(A) = \mathrm{tr}\left(\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}\right) = 1 + 5 = 6$$

The Elimination Method

The first method we will study for the solution of a set of equations is just ***an enlargement of the familiar method of eliminating one unknown*** between a pair of simultaneous equations

Generally called ***Gaussian elimination***

This method is the basic pattern of a large number of methods that are classified as ***direct methods***

Let us find the values for x_1, x_2, x_3 that solves the following set of equations

$$\begin{array}{rcrcrcr} 4x_1 & - & 2x_2 & + & x_3 & = & 15 \\ -3x_1 & - & x_2 & + & 4x_3 & = & 8 \\ x_1 & - & x_2 & + & 3x_3 & = & 13 \end{array}$$

First: Let us eliminate x_1

Multiplying the **first equation by 3** and **the second by 4** and adding

$$\begin{array}{rrcrcrcr} 3R_1 \longrightarrow & 12x_1 & - & 6x_2 & + & 3x_3 & = & 45 \\ 4R_2 \longrightarrow & -12x_1 & - & 4x_2 & + & 16x_3 & = & 32 \\ \hline 3R_1 + 4R_2 \longrightarrow & & - & 10x_2 & + & 19x_3 & = & 77 \end{array}$$

Multiplying the **first equation by -1** and the **third by 4** and adding

$$\begin{array}{rrcrcrcr} -1R_1 \longrightarrow & -4x_1 & + & 2x_2 & - & x_3 & = & -15 \\ 4R_3 \longrightarrow & 4x_1 & - & 4x_2 & + & 12x_3 & = & 52 \\ \hline -1R_1 + 4R_3 \longrightarrow & & - & 2x_2 & + & 11x_3 & = & 37 \end{array}$$

$$\begin{array}{crcrcr} - & 10x_2 & + & 19x_3 & = & 77 \\ - & 2x_2 & + & 11x_3 & = & 37 \end{array}$$

Let us find the values for x_1, x_2, x_3 that solves the following set of equations

$$\begin{aligned} 4x_1 &- 2x_2 + x_3 = 15 \\ -3x_1 &- x_2 + 4x_3 = 8 \\ x_1 &- x_2 + 3x_3 = 13 \end{aligned}$$

Second: Let us eliminate x_2

$$\begin{aligned} -10x_2 + 19x_3 &= 77 \\ -2x_2 + 11x_3 &= 37 \end{aligned}$$

Multiplying the **first by 2** and the **second by -10** and adding

$$\begin{aligned} 2R_1 \longrightarrow \quad & -20x_2 + 38x_3 = 154 \\ -10R_2 \longrightarrow \quad & +20x_2 - 110x_3 = -370 \\ 2R_1 + (-10)R_2 \longrightarrow \quad & -72x_3 = -216 \end{aligned}$$

Let us find the values for x_1, x_2, x_3 that solves the following set of equations

$$\begin{array}{rcrcrcr} 4x_1 & - & 2x_2 & + & x_3 & = & 15 \\ -3x_1 & - & x_2 & + & 4x_3 & = & 8 \\ x_1 & - & x_2 & + & 3x_3 & = & 13 \end{array}$$

Second: Let us eliminate x_2

$$\begin{array}{rrcrcr} - & 10x_2 & + & 19x_3 & = & 77 \\ - & 2x_2 & + & 11x_3 & = & 37 \end{array}$$

Multiplying the **first by 2** and the **second by -10** and adding

$$\begin{array}{lrrcrcr} 2R_1 \longrightarrow & - & 20x_2 & + & 38x_3 & = & 154 \\ -10R_2 \longrightarrow & + & 20x_2 & - & 110x_3 & = & -370 \\ \hline 2R_1 + (-10)R_2 \longrightarrow & & & - & 72x_3 & = & -216 \end{array}$$

Let us find the values for x_1, x_2, x_3 that solves the following set of equations

$$\begin{array}{rcrcrcr} 4x_1 & - & 2x_2 & + & x_3 & = & 15 \\ -3x_1 & - & x_2 & + & 4x_3 & = & 8 \\ x_1 & - & x_2 & + & 3x_3 & = & 13 \end{array}$$

$$\boxed{x_1 = 2}$$

Third: Back-substitute for finding the values for x_1, x_2, x_3

$$\begin{array}{rcrcrcr} - & 10x_2 & + & 19x_3 & = & 77 \\ - & 2x_2 & + & 11x_3 & = & 37 \end{array}$$

$$\boxed{x_2 = -2}$$

$$-72x_3 = -216$$

$$\boxed{x_3 = 3}$$

Now we present the same problem, solved in exactly the same way, but this time using **matrix notation**

$$\begin{array}{rcrcrcr} 4x_1 & - & 2x_2 & + & x_3 & = & 15 \\ -3x_1 & - & x_2 & + & 4x_3 & = & 8 \\ x_1 & - & x_2 & + & 3x_3 & = & 13 \end{array} \quad \xrightarrow{\boxed{Ax = b}} \quad \begin{bmatrix} 4 & -2 & 1 \\ -3 & -1 & 4 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 13 \end{bmatrix}$$

The arithmetic operations that we have performed affect only **the coefficients and the constant terms**, so we work with **the matrix of coefficients augmented** with the right-hand side vector

$$A|b = \left[\begin{array}{ccc:c} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{array}\right]$$

$$A|b = \left[\begin{array}{ccc:c} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{array}\right]$$

First: Let us eliminate x_1

Multiplying the first equation by 3 and the second by 4 and adding

Multiplying the first equation by -1 and the third by 4 and adding

$$\begin{array}{r} \\ 3R_1 + 4R_2 \rightarrow \\ (-1)R_1 + 4R_3 \rightarrow \end{array} \begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & -2 & 11 & 37 \end{bmatrix}$$

$$\begin{aligned} -10x_2 + 19x_3 &= 77 \\ -2x_2 + 11x_3 &= 37 \end{aligned}$$

$$\left[\begin{array}{ccc:c} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & -2 & 11 & 37 \end{array}\right]$$

Second: Let us eliminate x_2

Multiplying the first by 2 and the second by -10 and adding

$$\begin{array}{r} \\ \\ 2R_2 - 10R_3 \rightarrow \end{array} \begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & 0 & -72 & -216 \end{bmatrix}$$

$$-72x_3 = -216$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & 0 & -72 & -216 \end{array}\right]$$

$$\begin{aligned} 4x_1 - 2x_2 + x_3 &= 15 \\ -10x_2 + 19x_3 &= 77 \\ -72x_3 &= -216 \end{aligned}$$

Third: Back-substitute for finding the values for x_1, x_2, x_3

$$x_3 = \frac{-216}{-72} = 3$$

$$x_2 = \frac{(77 - 19(3))}{-10} = -2$$

$$x_1 = \frac{(15 - 1(3) - (-2)(-2))}{4} = 2$$

$$A|b = \left[\begin{array}{ccc|c} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & 0 & -72 & -216 \end{array}\right]$$

Transform to ***Upper-Triangular Matrix***

Elementary Row Transformations

Will not change the validity of the solution for the given set of equations

(1) We may multiply any row of the augmented coefficient matrix by a constant.
(2) We can add/subtract a multiple of one row to a multiple of any other row.
(3) We can interchange the order of any two rows

$$\begin{array}{ccccccc} 4x_1 & - & 2x_2 & + & x_3 & = & 15 \\ -3x_1 & - & x_2 & + & 4x_3 & = & 8 \\ x_1 & - & x_2 & + & 3x_3 & = & 13 \end{array} \Rightarrow \begin{bmatrix} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{bmatrix}$$

This process of Gaussian Elimination is good for hand calculations

What if we have a large set of equations?

the multiplications will give very large and unwieldy numbers that may overflow the computer's registers

Gaussian Elimination

1. Write the system of equations in **matrix form** $Ax = b$.
2. Form the **augmented matrix** $A|b$.
3. Perform elementary row operations to get zeros below the diagonal:
   - Multiply each element of the row by a non-zero constant
   - Switch two rows
   - Add (or subtract) a non-zero constant times a row to another row

Gaussian Elimination

$$A|b = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2m} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3m} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{4m} & b_4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nm} & b_n \end{bmatrix}$$

Column 1
"Elimination of x_1 from $R_2, R_3 \ldots R_n$"

Column 2
"Elimination of x_2 from $R_3, R_4 \ldots R_n$"

Column "*n*-1"
"Elimination of x_{n-1} from R_n"

Back Substitution
"Finding the values of of $x_1, x_2, \ldots, x_n$"

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2m} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3m} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{4m} & b_4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nm} & b_n \end{bmatrix}$$

Column 1

"Elimination of x_1 from $R_2, R_3 \ldots R_n$"

Pivot

$$\begin{matrix} \\ R_2 - \left({}^{a_{21}}/_{\boldsymbol{a_{11}}}\right)R_1 \longrightarrow \\ R_3 - \left({}^{a_{31}}/_{\boldsymbol{a_{11}}}\right)R_1 \longrightarrow \\ R_4 - \left({}^{a_{41}}/_{\boldsymbol{a_{11}}}\right)R_1 \longrightarrow \\ \\ R_n - \left({}^{a_{n1}}/_{\boldsymbol{a_{11}}}\right)R_1 \longrightarrow \end{matrix} \begin{bmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} & b_1 \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2m} & b_2 \\ 0 & a_{32} & a_{33} & a_{34} & \cdots & a_{3m} & b_3 \\ 0 & a_{42} & a_{43} & a_{44} & \cdots & a_{4m} & b_4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nm} & b_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} & b_1 \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2m} & b_2 \\ 0 & a_{32} & a_{33} & a_{34} & \cdots & a_{3m} & b_3 \\ 0 & a_{42} & a_{43} & a_{44} & \cdots & a_{4m} & b_4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nm} & b_n \end{bmatrix}$$

Column 2

"Elimination of x_2 from $R_3, R_4 \ldots R_n$"

Pivot

$$\begin{matrix} \\ \\ R_3 - \left({}^{a_{32}}/_{\boldsymbol{a_{22}}}\right)R_2 \longrightarrow \\ R_4 - \left({}^{a_{42}}/_{\boldsymbol{a_{22}}}\right)R_2 \longrightarrow \\ \\ R_n - \left({}^{a_{n2}}/_{\boldsymbol{a_{22}}}\right)R_2 \longrightarrow \end{matrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} & b_1 \\ 0 & \boldsymbol{a_{22}} & a_{23} & a_{24} & \cdots & a_{2m} & b_2 \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3m} & b_3 \\ 0 & 0 & a_{43} & a_{44} & \cdots & a_{4m} & b_4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & a_{n3} & a_{n4} & \cdots & a_{nm} & b_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} & b_1 \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2m} & b_2 \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3m} & b_3 \\ 0 & 0 & a_{43} & a_{44} & \cdots & a_{4m} & b_4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & a_{n3} & a_{n4} & \cdots & a_{nm} & b_n \end{bmatrix}$$

Column "*n*-1"

"Elimination of x_{n-1} from R_n"

$$R_n - \left({}^{a_{n(n-1)}}/\mathbf{a_{(n-1)(n-1)}} \right) R_{(n-1)} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} & b_1 \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2m} & b_2 \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3m} & b_3 \\ 0 & 0 & 0 & a_{44} & \cdots & a_{4m} & b_4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nm} & b_n \end{bmatrix}$$

Pivot

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} & b_1 \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2m} & b_2 \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3m} & b_3 \\ 0 & 0 & 0 & a_{44} & \cdots & a_{4m} & b_4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nm} & b_n \end{bmatrix}$$

Back Substitution

"Finding the values of of $x_1, x_2, \dots, x_n$"

$$a_{nm}x_n = b_n \Longrightarrow x_n = \frac{b_n}{a_{nm}}$$

$$a_{(n-1)(m-1)}x_{(n-1)} + a_{(n-1)m}x_n = b_{(n-1)}$$

Problem

We must guard **against dividing by zero**. Observe that zeros may **be created in the diagonal positions** even if they are not present in the original matrix of coefficients

Pivot

$$
\begin{matrix}
\\
\\
R_3 - (a_{32}/\boldsymbol{a_{22}})R_2 \longrightarrow \\
R_4 - (a_{42}/\boldsymbol{a_{22}})R_2 \longrightarrow \\
\\
R_n - (a_{n2}/\boldsymbol{a_{22}})R_2 \longrightarrow
\end{matrix}
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} & b_1 \\
0 & \boldsymbol{a_{22}} & a_{23} & a_{24} & \cdots & a_{2m} & b_2 \\
0 & a_{32} & a_{33} & a_{34} & \cdots & a_{3m} & b_3 \\
0 & a_{42} & a_{43} & a_{44} & \cdots & a_{4m} & b_4 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nm} & b_n
\end{bmatrix}
$$

Pivoting Operating

A useful strategy to avoid (if possible) such zero divisors is to rearrange the equations so as to put the coefficient of largest magnitude "*absolute value*" on the diagonal at each step

Complete Pivoting
May require both row and column interchanges. "*This is not frequently done*"

Partial Pivoting
Places a coefficient of larger magnitude "on the diagonal by **row interchanges only**

Guarantees a nonzero divisor if there is a solution to the set of equations

Must check for **possible row interchanges** before the elimination process of each column

Gaussian Elimination

$$A|b = \begin{bmatrix} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{bmatrix}$$

Column 1

"Elimination of x_1 from R_2, R_3"

$$\begin{matrix} \\ R_2 - \left(\frac{-3}{\mathbf{4}}\right)R_1 \rightarrow \\ R_3 - \left(\frac{1}{\mathbf{4}}\right)R_1 \rightarrow \end{matrix} \begin{bmatrix} \mathbf{4} & -2 & 1 & 15 \\ 0 & -2.5 & 4.75 & 19.25 \\ 0 & -0.5 & 2.75 & 9.25 \end{bmatrix}$$

Column 2

"Elimination of x_2 from R_3"

$$\begin{matrix} \\ \\ R_3 - \left(\frac{-0.5}{\mathbf{-2.5}}\right)R_2 \rightarrow \end{matrix} \begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & \mathbf{-2.5} & 4.75 & 19.25 \\ 0 & 0 & 1.8 & 5.40 \end{bmatrix}$$

Back Substitution

"Finding the values of of x_1, x_2, x_3"

$$1.8x_3 = 5.40 \Rightarrow \boldsymbol{x_3 = 3}$$

$$-2.5x_2 + 4.75x_3 = 19.25 \Rightarrow \boldsymbol{x_2 = 2}$$

$$4x_1 - 2x_2 + x_3 = 15 \Rightarrow \boldsymbol{x_1 = 4}$$

$$x = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

Gaussian Elimination

$$A|b = \begin{bmatrix} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{bmatrix}$$

Column 1

"Elimination of x_1 from R_2, R_3"

$$\begin{matrix} \\ R_2 - \left({}^{-3}/_{\mathbf{4}}\right)R_1 \rightarrow \\ R_3 - \left({}^{1}/_{\mathbf{4}}\right)R_1 \rightarrow \end{matrix} \begin{bmatrix} \mathbf{4} & -2 & 1 & 15 \\ (-0.75) & -2.5 & 4.75 & 19.25 \\ (0.25) & -0.5 & 2.75 & 9.25 \end{bmatrix}$$

Column 2

"Elimination of x_2 from R_3"

$$R_3 - \left({}^{-0.5}/_{\mathbf{-2.5}}\right)R_2 \rightarrow \begin{bmatrix} 4 & -2 & 1 & 15 \\ (-0.75) & \mathbf{-2.5} & 4.75 & 19.25 \\ (0.25) & (0.20) & 1.8 & 5.40 \end{bmatrix}$$

Back Substitution

"Finding the values of of x_1, x_2, x_3"

$$1.8x_3 = 5.40 \Rightarrow \boldsymbol{x_3 = 3}$$

$$-2.5x_2 + 4.75x_3 = 19.25 \Rightarrow \boldsymbol{x_2 = 2}$$

$$4x_1 - 2x_2 + x_3 = 15 \Rightarrow \boldsymbol{x_1 = 4}$$

$$x = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

Result from the final step

$$\begin{bmatrix} \mathbf{4} & \mathbf{-2} & \mathbf{1} & 15 \\ \color{red}{\mathbf{(-0.75)}} & \mathbf{-2.5} & \mathbf{4.75} & 19.25 \\ \color{red}{\mathbf{(0.25)}} & \color{red}{\mathbf{(0.20)}} & \mathbf{1.8} & 5.40 \end{bmatrix}$$

LU Decomposition

$$A = \begin{bmatrix} 4 & -2 & 1 \\ -3 & -1 & 4 \\ 1 & -1 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \color{red}{\mathbf{-0.75}} & 1 & 0 \\ \color{red}{\mathbf{0.25}} & \color{red}{\mathbf{0.20}} & 1 \end{bmatrix}}_{L} * \underbrace{\begin{bmatrix} \mathbf{4} & \mathbf{-2} & \mathbf{1} \\ 0 & \mathbf{-2.5} & \mathbf{4.75} \\ 0 & 0 & \mathbf{1.8} \end{bmatrix}}_{U}$$

L
Lower-Triangular

U
Upper-Triangular

Gaussian Elimination

$$\begin{array}{rcrcrcrcr} & & 2x_2 & + & & + & x_4 & = & 0 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & = & -2 \\ 4x_1 & - & 3x_2 & & & + & x_4 & = & -7 \\ 6x_1 & + & x_2 & - & 6x_3 & - & 5x_4 & = & 6 \end{array}$$

$$Ax = b$$

$$\begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 2 & 3 & 2 \\ 4 & -3 & 0 & 1 \\ 6 & 1 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -7 \\ 6 \end{bmatrix}$$

$$A|b = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix}$$

Gaussian Elimination

$$A|b = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix}$$

Column 1

"Elimination of x_1 from R_2, R_3"

Pivot = 0

Interchange row 1 with row 4

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Pivot = 6

$$\begin{matrix} \\ R_2 - (2/\mathbf{6})R_1 \rightarrow \\ R_3 - (4/\mathbf{6})R_1 \rightarrow \\ R_4 - (0/\mathbf{6})R_1 \rightarrow \end{matrix} \begin{bmatrix} \mathbf{6} & 1 & -6 & -5 & 6 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ (\mathbf{0.3333}) & 1.6667 & 5 & 3.6667 & -4 \\ (\mathbf{0.6667}) & -3.6667 & 4 & 4.3333 & -11 \\ (\mathbf{0}) & 2 & 0 & 1 & 0 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Column 2

"Elimination of x_2 from R_3, R_4"

Pivot = 1.6667

Interchange row 2 with row 3

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Pivot = -3.6667

$$\begin{matrix} \\ \\ R_3 - \left({}^{1.6667}/_{\mathbf{-3.6667}}\right) R_2 \rightarrow \\ R_4 - \left({}^{2}/_{\mathbf{-3.6667}}\right) R_2 \rightarrow \end{matrix} \begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & \mathbf{-3.6667} & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 2.1818 & 3.3636 & -5.9999 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ (\mathbf{0.3333}) & -3.6667 & 4 & 4.3333 & -11 \\ (\mathbf{0.6667}) & (\mathbf{-0.4545}) & 6.8182 & 5.6364 & -9.0001 \\ (\mathbf{0}) & (\mathbf{-0.5454}) & 2.1818 & 3.3636 & -5.9999 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 2.1818 & 3.3636 & -5.9999 \end{bmatrix}$$

Column 3

"Elimination of x_3 from R_4"

Pivot = 6.8182

$$R_4 - \left({}^{2.1818}/_{\mathbf{6.8182}}\right) R_3 \rightarrow \begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & \mathbf{6.8182} & 5.6364 & -9.0001 \\ 0 & 0 & 0 & 1.5600 & -3.1199 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ (\mathbf{0.3333}) & -3.6667 & 4 & 4.3333 & -11 \\ (\mathbf{0.6667}) & (\mathbf{-0.4545}) & 6.8182 & 5.6364 & -9.0001 \\ (\mathbf{0}) & (\mathbf{-0.5454}) & (\mathbf{0.32}) & 1.5600 & -3.1199 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 0 & 1.5600 & -3.1199 \end{bmatrix}$$

Back Substitution

$$x_4 = \frac{-3.1199}{1.5600} = -1.9999$$

$$x_3 = \frac{-9.0001 - 5.6364(-1.9999)}{6.8182} = 0.33325$$

$$x_2 = \frac{-11 - 4.3333(-1.9999) - 4(0.33325)}{-3.6667} = 1.0000$$

$$x_1 = \frac{6 - (-5)(-1.9999) - (-6)(0.33325) - (1)(1.0000)}{6} = -0.50000$$

$$x = \begin{bmatrix} 1.9999 \\ 0.33325 \\ 1.0000 \\ -0.50000 \end{bmatrix}$$

What is the LU Decomposition?

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ (\mathbf{0.3333}) & -3.6667 & 4 & 4.3333 & -11 \\ (\mathbf{0.6667}) & (\mathbf{-0.4545}) & 6.8182 & 5.6364 & -9.0001 \\ (\mathbf{0}) & (\mathbf{-0.5454}) & (\mathbf{0.32}) & 1.5600 & -3.1199 \end{bmatrix}$$

Augmented Matrix from last step

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ (\mathbf{0.3333}) & 1 & 0 & 0 \\ (\mathbf{0.6667}) & (\mathbf{-0.4545}) & 1 & 0 \\ (\mathbf{0}) & (\mathbf{-0.5454}) & (\mathbf{0.32}) & 1 \end{bmatrix}}_{\substack{L \\ \text{Lower-Triangular}}} * \underbrace{\begin{bmatrix} 6 & 1 & -6 & -5 \\ \mathbf{0} & -3.6667 & 4 & 4.3333 \\ \mathbf{0} & \mathbf{0} & 6.8182 & 5.6364 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1.5600 \end{bmatrix}}_{\substack{U \\ \text{Upper-Triangular}}}$$

What is the LU Decomposition?

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ (\mathbf{0.3333}) & -3.6667 & 4 & 4.3333 & -11 \\ (\mathbf{0.6667}) & (\mathbf{-0.4545}) & 6.8182 & 5.6364 & -9.0001 \\ (\mathbf{0}) & (\mathbf{-0.5454}) & (\mathbf{0.32}) & 1.5600 & -3.1199 \end{bmatrix}$$

Augmented Matrix from last step

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ (\mathbf{0.3333}) & 1 & 0 & 0 \\ (\mathbf{0.6667}) & (\mathbf{-0.4545}) & 1 & 0 \\ (\mathbf{0}) & (\mathbf{-0.5454}) & (\mathbf{0.32}) & 1 \end{bmatrix}}_{\substack{L \\ \text{Lower-Triangular}}} * \underbrace{\begin{bmatrix} 6 & 1 & -6 & -5 \\ \mathbf{0} & -3.6667 & 4 & 4.3333 \\ \mathbf{0} & \mathbf{0} & 6.8182 & 5.6364 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1.5600 \end{bmatrix}}_{\substack{U \\ \text{Upper-Triangular}}} \neq$$

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 2 & 3 & 2 \\ 4 & -3 & 0 & 1 \\ 6 & 1 & -6 & -5 \end{bmatrix}$$

Because we interchanged rows

$$A' = \begin{bmatrix} 6 & 1 & -6 & -5 \\ 4 & -3 & 0 & 1 \\ 2 & 2 & 3 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

What is the LU Decomposition?

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ (\mathbf{0.3333}) & -3.6667 & 4 & 4.3333 & -11 \\ (\mathbf{0.6667}) & (\mathbf{-0.4545}) & 6.8182 & 5.6364 & -9.0001 \\ (\mathbf{0}) & (\mathbf{-0.5454}) & (\mathbf{0.32}) & 1.5600 & -3.1199 \end{bmatrix}$$

Augmented Matrix from last step

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ (\mathbf{0.3333}) & 1 & 0 & 0 \\ (\mathbf{0.6667}) & (\mathbf{-0.4545}) & 1 & 0 \\ (\mathbf{0}) & (\mathbf{-0.5454}) & (\mathbf{0.32}) & 1 \end{bmatrix}}_{L} * \underbrace{\begin{bmatrix} 6 & 1 & -6 & -5 \\ \mathbf{0} & -3.6667 & 4 & 4.3333 \\ \mathbf{0} & \mathbf{0} & 6.8182 & 5.6364 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1.5600 \end{bmatrix}}_{U} = A' = \begin{bmatrix} 6 & 1 & -6 & -5 \\ 4 & -3 & 0 & 1 \\ 2 & 2 & 3 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

Lower-Triangular

Upper-Triangular

A permutation of A due to row interchanges

LU Decomposition

$$A = L * U$$

LU Decomposition is equal to A **If there is no row/column interchanges**

$$A' = L * U$$

LU Decomposition is equal to a permutation of A **If there is row/column interchanges**

Determinant Property

For square matrices A and B or equal size

$$\det(AB) = \det(A) * \det(B)$$

$$A = \begin{bmatrix} -5 & 3 \\ 7 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 5 \\ 2 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 11 & -7 \\ -3 & 47 \end{bmatrix}$$

$$\det(AB) = 496$$

$$\det(AB) = \begin{vmatrix} -5 & 3 \\ 7 & 2 \end{vmatrix} * \begin{vmatrix} -1 & 5 \\ 2 & 6 \end{vmatrix}$$

$$\det(AB) = (-31) * (-16) = 496$$

Determinant Property

If A is a **triangular matrix of order *n*** such that $a_{ij} = 0$ whenever $i > j$ or $i < j$

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

Lower-Triangular Matrix

If all the elements above the diagonal are zero

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 6 & 0 \\ -2 & 1 & -4 \end{bmatrix}$$

$$\det(L) = (1)(6)(-4) = -24$$

Upper-Triangular Matrix

If all the elements below the diagonal are zero

$$U = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(L) = (1)(-1)(1) = -1$$

LU Decomposition

$$A = L * U$$

LU Decomposition is equal to A **If there is no row/column interchanges**

$$\det(A) = \det(L * U) = \det(L) * \det(U)$$

$$\boxed{\det(A) = \det(L * U) = \det(U)}$$

$$A' = L * U$$

LU Decomposition is equal to a permutation of A **If there is row/column interchanges**

$$\det(A') = \det(L * U) = \det(L) * \det(U)$$

$$\boxed{\det(A') = \det(L * U) = \det(U)}$$

$$\boxed{\det(A) = (-1)^{m} * \det(L * U) = (-1)^{m}\det(U)}$$

***m** represents the number of row interchanges*

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ (\mathbf{0.3333}) & 1 & 0 & 0 \\ (\mathbf{0.6667}) & (\mathbf{-0.4545}) & 1 & 0 \\ (\mathbf{0}) & (\mathbf{-0.5454}) & (\mathbf{0.32}) & 1 \end{bmatrix}}_{\substack{L \\ \text{Lower-Triangular}}} * \underbrace{\begin{bmatrix} 6 & 1 & -6 & -5 \\ \mathbf{0} & -3.6667 & 4 & 4.3333 \\ \mathbf{0} & \mathbf{0} & 6.8182 & 5.6364 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1.5600 \end{bmatrix}}_{\substack{U \\ \text{Upper-Triangular}}} = \quad A' = \begin{bmatrix} 6 & 1 & -6 & -5 \\ 4 & -3 & 0 & 1 \\ 2 & 2 & 3 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

2-row Interchanges

$$\det(A') = \det(L * U) = \det(U) = (6)(-3.6667)(6.8182)(1.5600) = -234.0028$$

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 2 & 3 & 2 \\ 4 & -3 & 0 & 1 \\ 6 & 1 & -6 & -5 \end{bmatrix}$$

$$\det(A) = (-1)^m * \det(L * U) = (-1)^m * \det(U) = (-1)^2(6)(-3.6667)(6.8182)(1.5600) = -234.0028$$

Solve the system *Ax* = *b,* with multiple values of *b,* by **Gaussian elimination**

$$A = \begin{bmatrix} 3 & 2 & -1 & 2 \\ 1 & 4 & 0 & 2 \\ 2 & 1 & 2 & -1 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

$$b^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad b^{(2)} = \begin{bmatrix} -2 \\ 1 \\ 3 \\ 4 \end{bmatrix} \quad b^{(3)} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Form the Augmented matrix $A|b^{(1)}b^{(2)}b^{(3)}$

$$A|b^{(1)}b^{(2)}b^{(3)} = \begin{bmatrix} 3 & 2 & -1 & 2 & 0 & -2 & 2 \\ 1 & 4 & 0 & 2 & 0 & 1 & 2 \\ 2 & 1 & 2 & -1 & 1 & 3 & 0 \\ 1 & 1 & -1 & 3 & 0 & 4 & 0 \end{bmatrix}$$

Gaussian Elimination

$$A|b^{(1)}b^{(2)}b^{(3)} = \begin{bmatrix} 3 & 2 & -1 & 2 & 0 & -2 & 2 \\ 1 & 4 & 0 & 2 & 0 & 1 & 2 \\ 2 & 1 & 2 & -1 & 1 & 3 & 0 \\ 1 & 1 & -1 & 3 & 0 & 4 & 0 \end{bmatrix}$$

Column 1

"Elimination of x_1 from R_2, R_3, R_4"

Pivot = 3

$$\begin{matrix} \\ R_2 - (1/\mathbf{3})R_1 \rightarrow \\ R_3 - (2/\mathbf{3})R_1 \rightarrow \\ R_4 - (1/\mathbf{3})R_1 \rightarrow \end{matrix} \begin{bmatrix} \mathbf{3} & 2 & -1 & 2 & 0 & -2 & 2 \\ (\mathbf{0.333}) & 3.333 & 0.333 & 1.333 & 0 & 1.667 & 1.333 \\ (\mathbf{0.667}) & -0.333 & 2.667 & -2.333 & 1 & 4.333 & -1.333 \\ (\mathbf{0.333}) & 0.333 & -0.667 & 2.333 & 0 & 4.667 & -0.667 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 3 & 2 & -1 & 2 & 0 & -2 & 2 \\ (\mathbf{0.333}) & 3.333 & 0.333 & 1.333 & 0 & 1.667 & 1.333 \\ (\mathbf{0.667}) & -0.333 & 2.667 & -2.333 & 1 & 4.333 & -1.333 \\ (\mathbf{0.333}) & 0.333 & -0.667 & 2.333 & 0 & 4.667 & -0.667 \end{bmatrix}$$

Column 2

"Elimination of x_2 from R_3, R_4"

Pivot = 3.333

$$\begin{matrix} \\ \\ R_3 - \left({}^{-0.333}/_{\mathbf{3.333}}\right)R_2 \rightarrow \\ R_4 - \left({}^{0.333}/_{\mathbf{3.333}}\right)R_2 \rightarrow \end{matrix} \begin{bmatrix} 3 & 2 & -1 & 2 & 0 & -2 & 2 \\ (\mathbf{0.333}) & \mathbf{3.333} & 0.333 & 1.333 & 0 & 1.667 & 1.333 \\ (\mathbf{0.667}) & (\mathbf{-0.100}) & 2.700 & -2.200 & 1 & 4.500 & -1.200 \\ (\mathbf{0.333}) & (\mathbf{0.100}) & -0.700 & 2.200 & 0 & 4.500 & -0.800 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 3 & 2 & -1 & 2 & 0 & -2 & 2 \\ (\mathbf{0.333}) & 3.333 & 0.333 & 1.333 & 0 & 1.667 & 1.333 \\ (\mathbf{0.667}) & (\mathbf{-0.100}) & 2.700 & -2.200 & 1 & 4.500 & -1.200 \\ (\mathbf{0.333}) & (\mathbf{0.100}) & -0.700 & 2.200 & 0 & 4.500 & -0.800 \end{bmatrix}$$

Column 3

"Elimination of x_3 from R_4"

Pivot = 2.700

$$R_4 - \left({}^{-0.700}/_{\mathbf{2.700}}\right)R_3 \longrightarrow \begin{bmatrix} 3 & 2 & -1 & 2 & 0 & -2 & 2 \\ (\mathbf{0.333}) & 3.333 & 0.333 & 1.333 & 0 & 1.667 & 1.333 \\ (\mathbf{0.667}) & (\mathbf{-0.100}) & \mathbf{2.700} & -2.200 & 1 & 4.500 & -1.200 \\ (\mathbf{0.333}) & (\mathbf{0.100}) & (\mathbf{0.259}) & 1.630 & 0.259 & 5.667 & -1.111 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 3 & 2 & -1 & 2 & 0 & -2 & 2 \\ (\mathbf{0.333}) & 3.333 & 0.333 & 1.333 & 0 & 1.667 & 1.333 \\ (\mathbf{0.667}) & (\mathbf{-0.100}) & 2.700 & -2.200 & 1 & 4.500 & -1.200 \\ (\mathbf{0.333}) & (\mathbf{0.100}) & (\mathbf{0.259}) & 1.630 & 0.259 & 5.667 & -1.111 \end{bmatrix}$$

Back Substitution

$$x^{(1)} = \begin{bmatrix} 0.137 \\ -0.114 \\ 0.500 \\ 0.159 \end{bmatrix} \quad x^{(2)} = \begin{bmatrix} -0.591 \\ -1.340 \\ 4.500 \\ 3.477 \end{bmatrix} \quad x^{(3)} = \begin{bmatrix} 0.273 \\ 0.773 \\ -1.000 \\ -0.682 \end{bmatrix}$$

LU Decomposition

$$\begin{bmatrix} 3 & 2 & -1 & 2 & 0 & -2 & 2 \\ (\mathbf{0.333}) & 3.333 & 0.333 & 1.333 & 0 & 1.667 & 1.333 \\ (\mathbf{0.667}) & (\mathbf{-0.100}) & 2.700 & -2.200 & 1 & 4.500 & -1.200 \\ (\mathbf{0.333}) & (\mathbf{0.100}) & (\mathbf{0.259}) & 1.630 & 0.259 & 5.667 & -1.111 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ (\mathbf{0.333}) & 1 & 0 & 0 \\ (\mathbf{0.667}) & (\mathbf{-0.100}) & 1 & 0 \\ (\mathbf{0.333}) & (\mathbf{0.100}) & (\mathbf{0.259}) & 1 \end{bmatrix}}_{\substack{L \\ \text{Lower-Triangular}}} * \underbrace{\begin{bmatrix} 3 & 2 & -1 & 2 \\ \mathbf{0} & 3.333 & 0.333 & 1.333 \\ \mathbf{0} & \mathbf{0} & 2.700 & -2.200 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1.630 \end{bmatrix}}_{\substack{U \\ \text{Upper-Triangular}}} = A = \begin{bmatrix} 3 & 2 & -1 & 2 \\ 1 & 4 & 0 & 2 \\ 2 & 1 & 2 & -1 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

$$\det(A) = \det(L * U) = \det(L) * \det(U) = \det(U)$$

$$\det(A) = (3)(3.333)(2.700)(1.630) = 44.005599$$

Gaussian Elimination

Solves the System of Equations

Computes the determinant of a matrix efficiently

Provide us with the ***LU* decomposition** of the matrix of coefficients

Solve the system of equations using Gaussian Elimination

$$
\begin{array}{rcrcrcr}
 & + & 2x_2 & + & x_3 & = & -8 \\
x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\
-x_1 & + & x_2 & + & 2x_3 & = & 3
\end{array}
$$

Solve the system of equations using Gaussian Elimination

$$\begin{array}{rcrcrcr} & + & 2x_2 & + & x_3 & = & -8 \\ x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ -x_1 & + & x_2 & + & 2x_3 & = & 3 \end{array}$$

$$x = \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$$

Scaling

The operation of adjusting the coefficients of a set of equations so that they are all of the **same order of magnitude** to *avoid propagated round-off error*

$$\begin{bmatrix} 3 & 2 & 100 \\ -1 & 3 & 100 \\ 1 & 2 & -1 \end{bmatrix} x = \begin{bmatrix} 105 \\ 102 \\ 2 \end{bmatrix}$$

$$x_{exact} = \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \end{bmatrix}$$

Using Gaussian Elimination

With 3 digits precision

$$\begin{bmatrix} 3 & 2 & 100 & 105 \\ 0 & 3.66 & 133 & 137 \\ 0 & 0 & -82.6 & -82.7 \end{bmatrix}$$

$$x = \begin{bmatrix} 1.00 \\ 1.09 \\ 0.94 \end{bmatrix}$$

Partial Pivoting

By Scaling

dividing each row by the magnitude of the largest coefficient

$$\begin{matrix} \frac{R_1}{100} \rightarrow \\ \frac{R_2}{100} \rightarrow \\ \frac{R_3}{2} \rightarrow \end{matrix} \begin{bmatrix} 0.03 & 0.02 & 1.00 \\ -0.01 & 0.03 & 1.00 \\ 0.50 & 1.00 & -0.50 \end{bmatrix} x = \begin{bmatrix} 1.05 \\ 1.02 \\ 1.00 \end{bmatrix}$$

Using Gaussian Elimination

With 3 digits precision

$$\begin{bmatrix} 0.50 & 1.00 & -0.50 & 1.00 \\ 0 & 0.05 & 0.99 & 1.04 \\ 0 & 0 & 1.82 & 1.82 \end{bmatrix}$$

$$x = \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \end{bmatrix}$$

No Pivoting

Gauss-Jordan Elimination Method

"An extension to Gaussian Elimination"

The **diagonal elements** may all be made **ones as a first step** before creating zeros in their column; this **performs the divisions of the back-substitution phase at an earlier time**

Gauss-Jordan Elimination

$$A|b = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix}$$

Column 1

"Elimination of x_1 from R_2, R_3, R_4"

Pivot = 0

Interchange row 1 with row 4

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Pivot = 6

*Divide the **first row** by pivot "6"*

$$\frac{R_1}{\mathbf{6}} \rightarrow \begin{bmatrix} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Pivot = 1

$$\begin{matrix} \\ R_2 - (2/\mathbf{1})R_1 \rightarrow \\ R_3 - (4/\mathbf{1})R_1 \rightarrow \\ R_4 - (0/\mathbf{1})R_1 \rightarrow \end{matrix} \begin{bmatrix} \mathbf{1} & 0.1667 & -1 & -0.8333 & 1 \\ 0 & 1.6667 & 5 & 3.3667 & -4 \\ 0 & -3.6667 & 4 & 4.3334 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Gauss-Jordan Elimination

$$\begin{bmatrix} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & 1.6667 & 5 & 3.3667 & -4 \\ 0 & -3.6667 & 4 & 4.3334 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Column 2

"Elimination of x_2 from R_1, R_3, R_4"

Pivot = 1.6667

Interchange row 2 with row 3

$$\begin{bmatrix} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & -3.6667 & 4 & 4.3334 & -11 \\ 0 & 1.6667 & 5 & 3.3667 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Pivot = -3.6667

*Divide the **second row** by pivot "-3.6667"*

$$\frac{R_2}{-3.6667} \longrightarrow \begin{bmatrix} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 1.6667 & 5 & 3.3667 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Pivot = 1

$$\begin{matrix} R_1 - \left({}^{0.1667}/_{1}\right)R_2 \longrightarrow \\ \\ R_3 - \left({}^{1.6667}/_{1}\right)R_2 \longrightarrow \\ R_4 - \left({}^{2}/_{1}\right)R_2 \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & -0.8182 & -0.6364 & 0.5 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 0 & 6.8182 & 5.6364 & -9 \\ 0 & 0 & 2.1818 & 3.3636 & -6 \end{bmatrix}$$

Elimination is done above and below diagonal

Gauss-Jordan Elimination

$$\begin{bmatrix} 1 & 0 & -0.8182 & -0.6364 & 0.5 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 0 & 6.8182 & 5.6364 & -9 \\ 0 & 0 & 2.1818 & 3.3636 & -6 \end{bmatrix}$$

Column 3

"Elimination of x_3 from R_1, R_2, R_4"

Pivot = 6.8182

Divide the ***third row*** *by pivot "6.8182"*

$$\frac{R_3}{\mathbf{6.8182}} \longrightarrow \begin{bmatrix} 1 & 0 & -0.8182 & -0.6364 & 0.5 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 0 & 1 & 0.8267 & -1.32 \\ 0 & 0 & 2.1818 & 3.3636 & -6 \end{bmatrix}$$

Pivot = 1

$$\begin{matrix} R_1 - \left({}^{-0.8182}/_{\mathbf{1}}\right)R_3 \longrightarrow \\ R_2 - \left({}^{-1.0909}/_{\mathbf{1}}\right)R_3 \longrightarrow \\ \\ R_4 - \left({}^{2.1818}/_{\mathbf{1}}\right)R_3 \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0.04 & -0.58 \\ 0 & 1 & 0 & -0.280 & 1.56 \\ 0 & 0 & \mathbf{1} & 0.8267 & -1.32 \\ 0 & 0 & 0 & 1.5599 & -3.12 \end{bmatrix}$$

Elimination is done above and below diagonal

Gauss-Jordan Elimination

$$\begin{bmatrix} 1 & 0 & 0 & 0.04 & -0.58 \\ 0 & 1 & 0 & -0.280 & 1.56 \\ 0 & 0 & 1 & 0.8267 & -1.32 \\ 0 & 0 & 0 & 1.5599 & -3.12 \end{bmatrix}$$

Column 4

"Elimination of x_4 from R_1, R_2, R_3"

Pivot = 1.5599

*Divide the **fourth row** by pivot "1.5599"*

$$\frac{R_3}{\mathbf{1.5599}} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0.04 & -0.58 \\ 0 & 1 & 0 & -0.280 & 1.56 \\ 0 & 0 & 1 & 0.8267 & -1.32 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Pivot = 1

$$\begin{matrix} R_1 - \left({}^{0.04}/_{\mathbf{1}}\right)R_4 \rightarrow \\ R_2 - \left({}^{-0.280}/_{\mathbf{1}}\right)R_4 \rightarrow \\ R_3 - \left({}^{0.8267}/_{\mathbf{1}}\right)R_4 \rightarrow \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 & 1.0001 \\ 0 & 0 & 1 & 0 & 0.3333 \\ 0 & 0 & 0 & \mathbf{1} & -2 \end{bmatrix}$$

Elimination is done above and below diagonal

Gauss-Jordan Elimination

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 & 1.0001 \\ 0 & 0 & 1 & 0 & 0.3333 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\boxed{x = \begin{bmatrix} -0.5 \\ 1.0001 \\ 0.3333 \\ -2 \end{bmatrix}}$$

Solve the system of equations using Gauss-Jordan Elimination

$$\begin{array}{rcrcrcr} & + & 2x_2 & + & x_3 & = & -8 \\ x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ -x_1 & + & x_2 & + & 2x_3 & = & 3 \end{array}$$

$$x = \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$$

Gauss-Jordan Elimination **requires 50% more operations** than Gaussian Elimination

Gaussian Elimination		Gauss-Jordan Elimination
$O(\frac{n^3}{3} + n^2 - \frac{n}{3})$	$<$	$O(\frac{n^3}{2} + n^2 - \frac{7n}{2} + 2)$

Pathology in Linear Systems

For arbitrary system of linear equations $Ax = b$,

one of the following cases must hold:

Consistent System
- It has a unique solution
- It has no unique solution "infinite solutions"

Inconsistent System
- It has no solution

What are the possible different kinds of equation in a linear system?

Type 1 Equation

An equation that provides information about the unknowns *without any repetition or conflict "inconsistency"* with any previous information provided by previous equation in the system

Type 2 Equation

Redundant equation that provides information about the unknowns *that has been provided* by previous equations in the system

Type 3 Equation

An equation that provides information about the unknowns that is inconsistent with previous information provided by previous equations in the system

$x + y = 2$	*Type 1 Equation*
$x - 2y = 3$	*Type 1 Equation*
$2x + 2y = 4$	*Type 2 Equation*
$2x - y = 1$	*Type 3 Equation*

$n \times n$ linear system has n variables and and n equations

For arbitrary system of linear equations $Ax = b$,

one of the following cases must hold:

Consistent System

- **It has a unique solution**

  If the number of type 1 equations is equal to the the number of unknowns

- **It has no unique solution "infinite solutions"**

  If the number of type 1 equations is less than the number of unknowns

Inconsistent System

- **It has no solution**

  If there is at least one type 3 equation regardless of the number of unknowns or equations

For arbitrary system of linear equations $Ax = b$, *if the number of equations is less than the number of unknowns then* **its impossible to have a unique solution**

Maybe infinite solutions OR no solution

For arbitrary system of linear equations $Ax = b$, *if the number of equations is equal to the number of unknowns then* **its not necessary for the system to have a unique solution**

Except if all equations were type 1 equations

For an arbitrary system of linear equations $Ax = b$, and its augmented matrix $A|b$, there are 3 possibilities:

CASE 1:

$A|b$ → Gaussian Elimination →

$$\left[\begin{array}{cccc|c} \neq 0 & & & & \\ 0 & \neq 0 & & & \\ 0 & 0 & \neq 0 & & \\ 0 & 0 & 0 & \neq 0 & \end{array}\right]$$

Full Rank Coefficient Matrix
"n-Rank"

Nonsingular
Coefficient Matrix

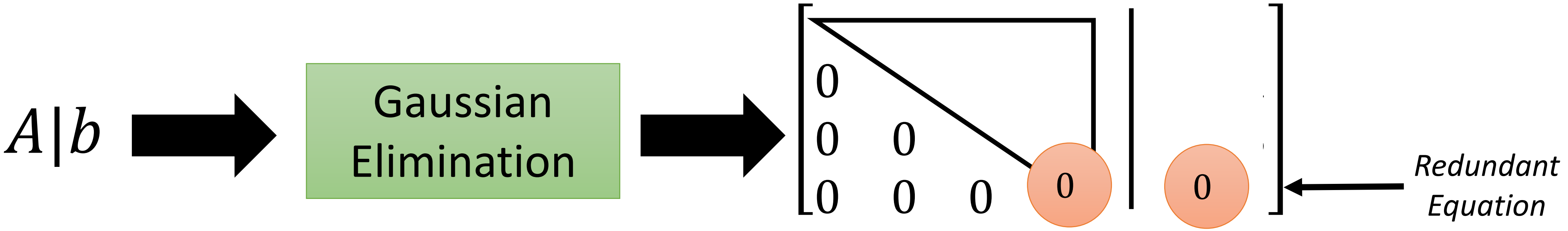
$\det(A) \neq 0$

Rank(A)*= Number of non-zero rows in the triangulated coefficient matrix*

Unique Solution

For an arbitrary system of linear equations $Ax = b$, and its augmented matrix $A|b$, there are 3 possibilities:

CASE 2:



If one or more zeros occur on the final diagonal, there is no unique solution and the set will be consistent (and have redundancy) if back-substitution gives (0/0) indeterminate forms

Singular *Coefficient Matrix*

$\det(A) = 0$

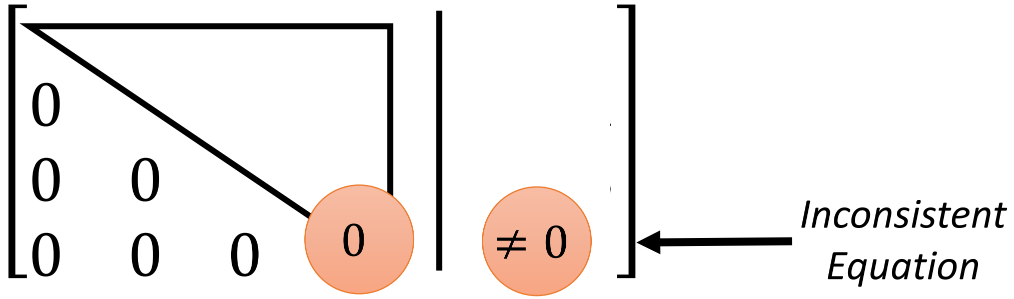
Rank(A)*= Number of non-zero rows in the triangulated coefficient matrix*

No Unique Solution

For an arbitrary system of linear equations $Ax = b$, and its augmented matrix $A|b$, there are 3 possibilities:

CASE 3:

$A|b$ → Gaussian Elimination →



If one or more zeros occur on the final diagonal, there is no solution and the set will be inconsistent if back-substitution divided a nonzero term by zero

Singular *Coefficient Matrix*

$\det(A) = 0$

Rank(A)*= Number of non-zero rows in the triangulated coefficient matrix*

No Solution

Singular Matrix	Nonsingular Matrix
The coefficient matrix is singular	The coefficient matrix is nonsingular
The set of equations with these coefficients has no unique solution	The set of equations with these coefficients has a unique solution
Gaussian elimination cannot avoid a zero on the diagonal	Gaussian elimination proceeds without a zero on the diagonal
The rank of the coefficient matrix is less than n	The rank of the coefficient matrix is equal to n
The coefficient matrix has a zero determinant	The coefficient matrix has a nonzero determinant
The coefficient matrix has no inverse	The coefficient matrix has an inverse

Find **Rank**(A), then determine whether the following systems: $Ax^{(1)} = b^{(1)}, Ax^{(2)} = b^{(2)}$ have unique solution, infinity many solutions or no solution

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \\ -1 & -14 & 11 \end{bmatrix}, b^{(1)} = \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix}, b^{(2)} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}$$

$$A|b^{(1)}|b^{(2)} = \begin{bmatrix} 1 & -2 & 3 & 5 & 5 \\ 2 & 4 & -1 & 7 & 7 \\ -1 & -14 & 11 & 2 & 1 \end{bmatrix}$$

→ Gaussian Elimination →

$$\begin{bmatrix} 1 & -2 & 3 & 5 & 5 \\ 0 & 8 & -7 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Rank(A) = 2 < 3

"Singular Matrix"

For any b, there will be no unique solution

$Ax^{(1)} = b^{(1)}$	*No Solution*
$Ax^{(2)} = b^{(2)}$	*Infinite "Many" Solutions*

Matrix Determinant

$$A = L * U$$

LU Decomposition is equal to A **If there is no row/column interchanges**

$$\det(A) = \det(L * U) = \det(L) * \det(U)$$

$$\boxed{\det(A) = \det(L * U) = \det(U)}$$

$$A' = L * U$$

LU Decomposition is equal to a permutation of A **If there is row/column interchanges**

$$\det(A') = \det(L * U) = \det(L) * \det(U)$$

$$\boxed{\det(A') = \det(L * U) = \det(U)}$$

$$\boxed{\det(A) = (-1)^m * \det(L * U) = (-1)^m \det(U)}$$

m *represents the number of row interchanges*

$$\begin{vmatrix} 1 & 4 & -2 & 3 \\ 2 & 2 & 0 & 4 \\ 3 & 0 & -1 & 2 \\ 1 & 2 & 2 & -3 \end{vmatrix} \rightarrow \text{Gaussian Elimination} \rightarrow \begin{vmatrix} 1 & 4 & -2 & 3 \\ 0 & -6 & 4 & -2 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & -8 \end{vmatrix}$$

With no Row Interchanges

$$\boxed{\det(A) = (1)(-6)(-3)(-8) = \mathbf{-144}}$$

$$\begin{vmatrix} 1 & 4 & -2 & 3 \\ 2 & 2 & 0 & 4 \\ 3 & 0 & -1 & 2 \\ 1 & 2 & 2 & -3 \end{vmatrix} \rightarrow \text{Gaussian Elimination} \rightarrow \begin{vmatrix} 3 & 0 & -1 & 2 \\ 0 & 4 & -1.677 & 2.333 \\ 0 & 0 & 3.167 & -4.833 \\ 0 & 0 & 0 & 3.789 \end{vmatrix}$$

With 3 Row Interchanges

$$\boxed{\det(A) = (-1)^3(3)(4)(3.167)(3.789) = \mathbf{-144}}$$

Matrix Inverse

While division of matrices is not defined, the *matrix inverse gives the equivalent result*

Not all square matrices have an inverse

Singular matrices do ***not*** *have an inverse*

If the product of ***two square matrices*** *is the* **identity matrix**, *the matrices are said to be* ***inverses***

$$AB = \boldsymbol{I} \Longrightarrow B = A^{-1} \text{ and } A = B^{-1}$$

How to compute a Matrix Inverse?

Gaussian Elimination

Augment the given matrix *A* with the **identity matrix of the same order**

Reduces the augmented matrix to the **identity matrix** by elementary row transformations

Apply back-substitution to last right half of the augmented matrix, **the inverse of the original stands as the right half**

Gauss-Jordan Elimination

Augment the given matrix *A* with the **identity matrix of the same order**

Reduces the augmented matrix to the **identity matrix** by elementary row transformations

When the identity matrix stands as the left half of the augmented matrix, **the inverse of the original stands as the right half**

$$\text{Find } A^{-1} \text{ for } A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

Gaussian Elimination

$$A|\boldsymbol{I} = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 1 & 0 & 1 & 0 \\ (\mathbf{0.333}) & -1 & 1.667 & 1 & -0.333 & 0 \\ (\mathbf{0.333}) & (\mathbf{0}) & 1.667 & 0 & -0.333 & 1 \end{bmatrix}$$

Apply back-substitution for the last three columns

$$\begin{bmatrix} 3 & 0 & 1 & 0 & 1 & 0 \\ (\mathbf{0.333}) & -1 & 1.667 & 1 & -0.333 & 0 \\ (\mathbf{0.333}) & (\mathbf{0}) & 1.667 & 0 & -0.333 & 1 \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad x^{(2)} = \begin{bmatrix} 0.4 \\ 0 \\ -0.2 \end{bmatrix} \quad x^{(3)} = \begin{bmatrix} -0.2 \\ 1 \\ 0.6 \end{bmatrix}$$

Gauss-Jordan Elimination

$$A|\boldsymbol{I} = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & \frac{2}{5} & -\frac{1}{5} \\ -1 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

We confirm the fact that we have found the inverse by multiplication

$$AA^{-1} = \boldsymbol{I}$$

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} * \begin{bmatrix} 0 & \frac{2}{5} & -\frac{1}{5} \\ -1 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

Iterative Methods

Preferred over the direct methods especially when the coefficient matrix is sparse (has many zeros), these methods may be more rapid and economical regarding memory requirements

For hand computation they have the distinct advantage that they are self-correcting if an error is made

Jacobi Method

Gauss-Seidel Method

Jacobi Method

$$\begin{aligned} 6x_1 &- 2x_2 + x_3 = 11 \\ x_1 &+ 2x_2 - 5x_3 = -1 \\ -2x_1 &+ 7x_2 + 2x_3 = 5 \end{aligned}$$

$$x_{\text{actual}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Begin by solving each equation for one of the variables, choosing, *when possible*, to solve for the **variable with the largest coefficient**

Start with some initial approximation to the value of the variables. (Or zero if no better initial estimates are at hand)

Substituting these approximations into the right-hand sides of the set of equations generates new approximations that, we hope, are closer to the true value

The new values are substituted in the right-hand sides to generate a second approximation, and the process is repeated until successive values of **each variable are sufficiently alike**

From 1st Equation $\Rightarrow x_1^{(n+1)} = 1.8333 + 0.3333x_2^{(n)} - 0.1667x_3^{(n)}$

From 3rd Equation $\Rightarrow x_2^{(n+1)} = 0.7143 + 0.2857x_1^{(n)} - 0.2857x_3^{(n)}$

From 2nd Equation $\Rightarrow x_3^{(n+1)} = 0.2000 + 0.2000x_1^{(n)} + 0.4000x_2^{(n)}$

Starting with an initial vector of $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

	First	Second	Third	Fourth	Fifth	Sixth	Seventh	Eighth	Ninth
x_1	0	1.833	2.038	2.085	2.004	1.994	1.996	2.000	**2.000**
x_2	0	0.714	1.181	1.053	1.001	0.990	0.998	1.000	**1.000**
x_3	0	0.200	0.852	1.080	1.038	1.001	0.995	0.998	**1.000**

How to compute the new set of equation *programmatically*?

$$\begin{array}{rcrcrcr} 6x_1 & - & 2x_2 & + & x_3 & = & 11 \\ x_1 & + & 2x_2 & - & 5x_3 & = & -1 \\ -2x_1 & + & 7x_2 & + & 2x_3 & = & 5 \end{array}$$

$$\boxed{\begin{array}{lcccccc} x_1^{(n+1)} & = & 1.8333 & + & 0.3333x_2^{(n)} & - & 0.1667x_3^{(n)} \\ x_2^{(n+1)} & = & 0.7143 & + & 0.2857x_1^{(n)} & - & 0.2857x_3^{(n)} \\ x_3^{(n+1)} & = & 0.2000 & + & 0.2000x_1^{(n)} & + & 0.4000x_2^{(n)} \end{array}}$$

$$Ax = b$$

$$\begin{bmatrix} 6 & -2 & 1 \\ -2 & 7 & 2 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ -1 \end{bmatrix}$$

$$\begin{array}{ccccccc} A & = & L & + & D & + & U \end{array}$$

$$\begin{bmatrix} 6 & -2 & 1 \\ -2 & 7 & 2 \\ 1 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -5 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Ax = (L + D + U)x = b \implies Dx = -(L + U)x + b$$

$$\boxed{x = -D^{-1}(L + U)x + D^{-1}b}$$

$$x^{(n+1)} = -D^{-1}(L + U)x^{(n)} + D^{-1}b$$

$$x^{(n+1)} = -Bx^{(n)} + b'$$

$$B = D^{-1}(L + U) = \begin{bmatrix} 0 & -0.3333 & 0.1667 \\ -0.2857 & 0 & 0.2857 \\ -0.2000 & -0.4000 & 0 \end{bmatrix}$$

$$b' = D^{-1}b = \begin{bmatrix} 1.8333 \\ 0.7143 \\ 0.2000 \end{bmatrix}$$

Fixed-Point Iteration Method

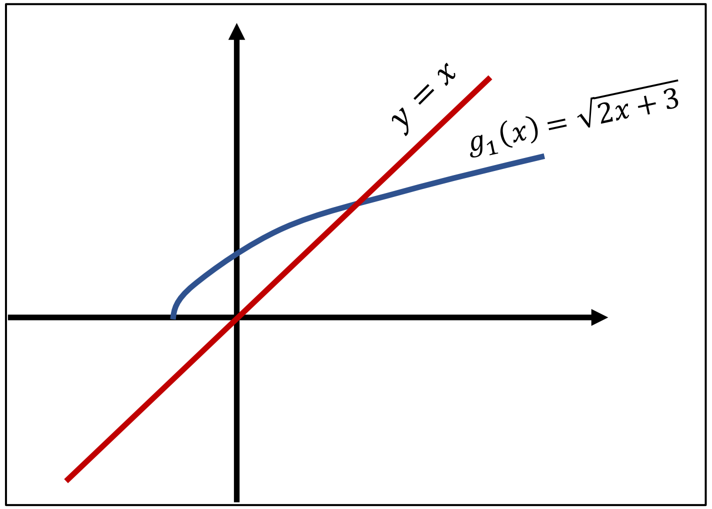
$$x_{n+1} = g(x_n) \qquad n = 0,1,2,3,\ldots$$

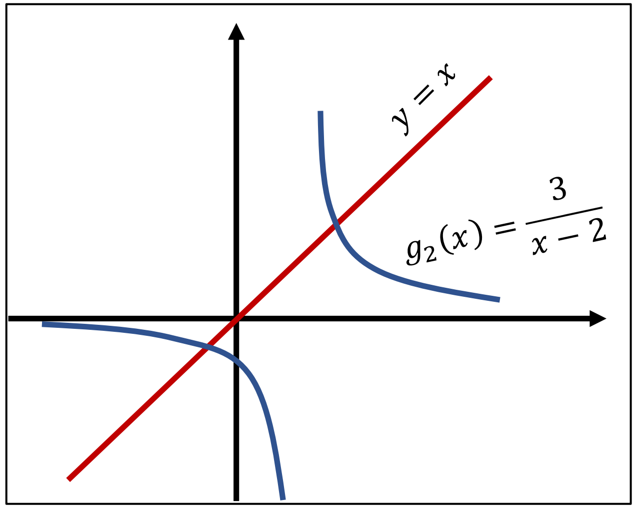
$$f(x) = x^2 - 2x - 3 = 0$$

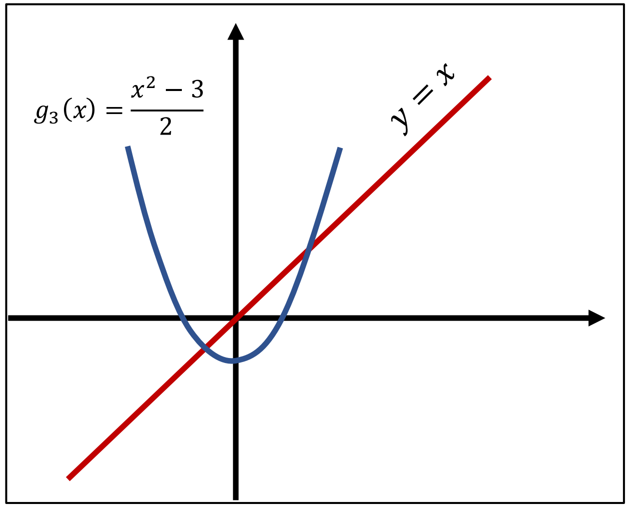
$g_1(x) = \sqrt{2x+3}$

$g_2(x) = \dfrac{3}{x-2}$

$g_3(x) = \dfrac{x^2-3}{2}$







Jacobi method for a *set of equations* is exactly the same as the **fixed-point iteration** method for a *single equation*

Jacobi Method

$$x^{(n+1)} = G\left(x^{(n)}\right) = b' - Bx^{(n)}$$
$$n = 0,1,2,3,\dots$$

Fixed-Point Iteration Method

$$x_{n+1} = g(x_n)$$
$$n = 0,1,2,3,\dots$$

Gauss-Seidel Method

$$\begin{aligned} 6x_1 &- 2x_2 &+ x_3 &= 11 \\ x_1 &+ 2x_2 &- 5x_3 &= -1 \\ -2x_1 &+ 7x_2 &+ 2x_3 &= 5 \end{aligned}$$

$$x_{\text{actual}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Begin by solving each equation for one of the variables, choosing, *when possible*, to solve for the **variable with the largest coefficient**

From 1st Equation $\Longrightarrow x_1^{(n+1)} = 1.8333 + 0.3333x_2^{(n)} - 0.1667x_3^{(n)}$

From 3rd Equation $\Longrightarrow x_2^{(n+1)} = 0.7143 + 0.2857\boldsymbol{x_1^{(n+1)}} - 0.2857x_3^{(n)}$

From 2nd Equation $\Longrightarrow x_3^{(n+1)} = 0.2000 + 0.2000\boldsymbol{x_1^{(n+1)}} + 0.4000\boldsymbol{x_2^{(n+1)}}$

Start with some initial approximation to the value of the variables. (Or zero if no better initial estimates are at hand)

Starting with an initial vector of $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Proceed by improving each x-value in turn, **always using the most recent approximations** to the values of the other variables

The process is repeated until successive values of **each variable are sufficiently alike**

	First	Second	Third	Fourth	Fifth	Sixth
x_1	0	1.833	2.069	1.998	1.999	**2.000**
x_2	0	1.238	1.002	1.053	1.000	**1.000**
x_3	0	1.062	1.015	1.080	1.000	**1.000**

The rate of convergence is more rapid!