CPE 310: Numerical Analysis for Engineers

Chapter 5: Numerical Solution of Ordinary Differential Equations

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Ordinary differential equations is essential for modelling many physical situations. These equations have also demonstrated their usefulness in many fields.

The basic problem to be solved

$$y' = f(x, y), \qquad y(x_0) = y_0$$

Taylor-Series Method

Euler and Modified Euler Methods

Runge-Kutta Methods

The Taylor-Series Method

It is not strictly a numerical method, but it is sometimes used in conjunction with numerical schemes

The Taylor-Series Method

We develop the relation between y and x by finding the coefficients of the Taylor series in which we expand y about the point $x = x_0$

$$\boxed{y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \cdots}$$

"Initial Condition"

If we let $\boldsymbol{x - x_0 = h}$, we can write the series as:

$$y(x) = y(x_0) + y'(x_0)h + \frac{y''(x_0)}{2!}h^2 + \frac{y'''(x_0)}{3!}h^3 + \cdots$$

If the expansion or the "initial condition" is about the point $x_0 = 0$, the Taylor series is actually the **Maclaurin series**

Example

Given the following initial condition, find the solution for y using Taylor-series method using $n = 4$

$$\frac{dy}{dx} = y' = -2x - y, \qquad y(0) = -1$$

$$y(x) \approx y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \frac{y^{iv}(x_0)}{4!}(x - x_0)^4$$

From the given equations: $x_0 = 0,\ y(x_0) = -1$

We get the second and higher derivatives by successively differentiating the equation for the first derivative. Each of these derivatives is evaluated corresponding to $x = 0$ to get the various derivatives

$$y'(x) = -2x - y, \qquad y'(x_0) = -2(x_0) - y(x_0) = -2(0) - (-1) = 1$$

$$y''(x) = -2 - y', \qquad y''(x_0) = -2 - y'(x_0) = -2 - 1 = -3$$

$$y'''(x) = -y'', \qquad y'''(x_0) = -y''(x_0) = -(-3) = 3$$

$$y^{iv}(x) = -y''', \qquad y^{iv}(x_0) = -y'''(x_0) = -(3) = -3$$

If we let $\boldsymbol{x - x_0 = h}$, we can write the series as:

$$y(h) = -1 + h - 1.5h^2 + 0.5h^3 - 0.125h^4 + \text{error}$$

$$y(h) = -1 + h - 1.5h^2 + 0.5h^3 - 0.125h^4 + \text{error}$$

$$y(x) = -3e^{-x} - 2x + 2$$

$\boldsymbol{x}$	$\boldsymbol{y(x - x_0)}$	$\boldsymbol{y_{\text{actual}}(x)}$
0.0	−1.00000	−1.00000
0.1	−0.91451	−0.91451
0.2	−0.85620	−0.85619
0.3	−0.82251	−0.82245
0.4	−0.81120	−0.81096
0.5	−0.82031	−0.81959

The Taylor-Series Method

The Taylor series is easily applied to a higher-order equation

Given the following initial condition:

$$\boxed{y'' = 3 + x - y^2, \qquad y(0) = 1, \qquad y'(0) = -2}$$

$$y(x) = y(x_0) + y'(x_0)h + \frac{y''(x_0)}{2!}h^2 + \frac{y'''(x_0)}{3!}h^3 + \frac{y^{iv}(x_0)}{4!}h^4 + O(h^5)$$

$$y''(x) = -3 + x - y^2, \qquad y''(x_0) = -3 + x_0 - \left(y(x_0)\right)^2 = -3 + 0 - 1^2 = -4$$

$$y'''(x) = 1 - 2yy', \qquad y'''(x_0) = 1 - 2(1)(-2) = 5$$

$$y^{iv}(x) = -2yy'' - 2y'y', \qquad y^{iv}(x_0) = -2y(x_0)y''(x_0) - 2y'(x_0)y'(x_0) = -2(1)(5) - 2(-2)(-2) = -18$$

$$y(h) = 1 - 2h \; - 2h^2 + 0.833h^3 - 0.75h^4 + \text{error}$$

The Taylor-series method may be **awkward** to apply if the derivatives **become complicated** and in this case the **error is difficult to determine**

The error in a Taylor series will be **small** if the step size $\boldsymbol{h}$ **is small**. If h is small enough, we may only need a few terms of the Taylor-series expansion for **good accuracy**

Euler and Modified Euler Methods

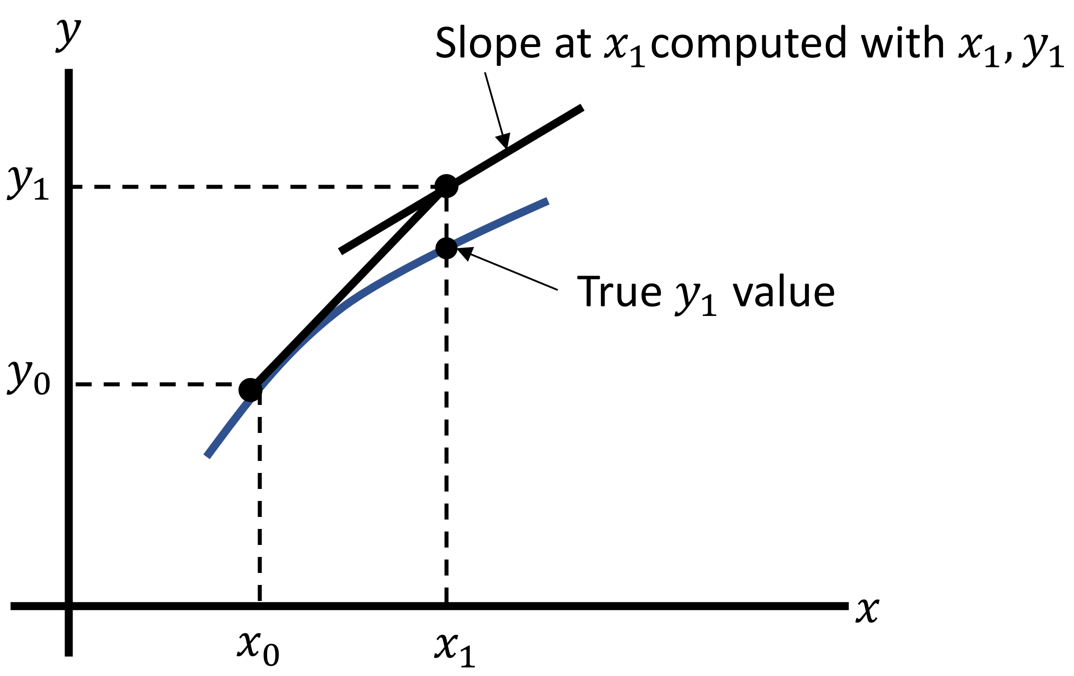
Simpler Euler Method

The Euler method uses only the first two terms of the Taylor series for first-order differential equations

$$\boxed{y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2}$$

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \boldsymbol{O(h^2)}$$

$$y_{n+1} = y_n + hy_n' + \boldsymbol{O(h^2)}$$



Example

Use Euler method to solve the following with $h = 0.1$ to get $y(0.4)$

$$\frac{dy}{dx} = y' = -2x - y, \qquad y(0) = -1$$

$$y_{actual}(0.4) = -0.81096$$

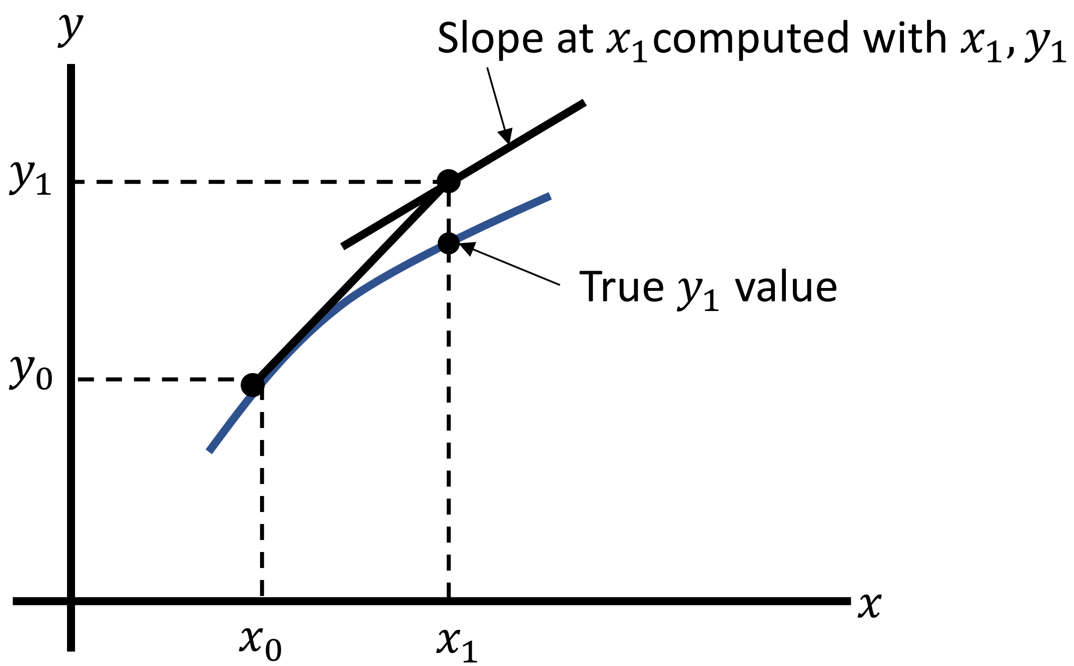
$$y_{n+1} \approx y_n + hy'_n$$

x_n	y_n	y'_n	hy'_n	y_{n+1}
0.0	-1.00000	1.00000	0.10000	-0.90000
0.1	-0.90000	0.70000	0.07000	-0.83000
0.2	-0.83000	0.43000	0.04300	-0.78700
0.3	-0.78700	0.18700	0.01870	$\mathbf{-0.76830}$
0.4	$\mathbf{-0.76830}$			

Actual Error is: $y_{actual}(0.4) - y_{appro}(0.4) = -0.04266$

In the simple Euler method, we use the slope at the **beginning of the interval** to determine the increment to the function.

This technique would be correct only if the function was ***linear***

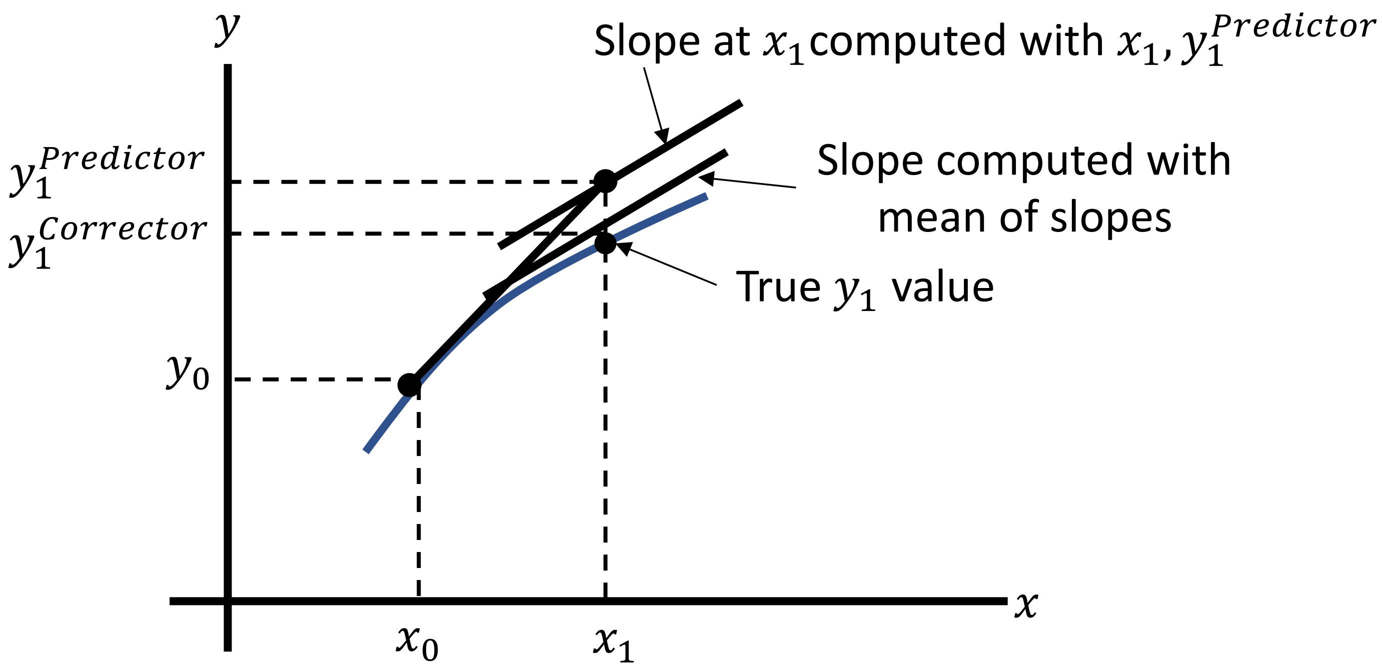


Modified Euler Method

We use the correct average slope within the interval. This can be approximated by the mean of the slopes at both ends of the interval.

$$\boxed{y_{n+1}^{\text{Predictor}} = y_n + h y_n'}$$

$$y_{n+1}^{\text{Corrector}} = y_n + h \frac{y_n' + y'^{\text{Predictor}}_{n+1}}{2}$$



Example

Use Modified Euler method to solve the following with $h = 0.1$ to get $y(0.5)$

$$\frac{dy}{dx} = y' = -2x - y, \qquad y(0) = -1$$

$$y_{actual}(0.5) = -0.8196$$

$$y_{n+1}^{\text{Predictor}} = y_n + h y'_n$$

$$y_{n+1}^{\text{Corrector}} = y_n + h \frac{y'_n + y'^{\text{Predictor}}_{n+1}}{2}$$

$\boldsymbol{x_n}$	$\boldsymbol{y_n}$	$y_{n+1}^{\text{Predictor}}$	$y_{n+1}^{\text{Corrector}}$
0.0	-1.0000	-0.9000	-0.9150
0.1	-0.9150	-0.8453	-0.8571
0.2	-0.8571	-0.8114	-0.8237
0.3	-0.8237	-0.8013	-0.8124
0.4	-0.8124	-0.8112	$\mathbf{-0.8212}$
0.5	$\mathbf{-0.8212}$		

Runge-Kutta Methods

While we can improve the accuracy of previous methods by taking smaller step sizes, much greater accuracy can be obtained more efficiently by Runge-Kutta methods

Second-Order Runge-Kutta methods

Similar to Modified Euler method

Fourth-Order Runge-Kutta methods

2^{nd} Order Runge-Kutta Methods

$$y_{n+1} = y_n + (a\boldsymbol{k_1} + b\boldsymbol{k_2})h$$

$$\boldsymbol{k_1} = f(x_n, y_n) \qquad \boldsymbol{k_2} = f(x_n + \alpha h, y_n + \beta \boldsymbol{k_1} h)$$

$$a + b = 1 \qquad \alpha b = \frac{1}{2} \qquad \beta b = \frac{1}{2}$$

3 Equations with 4 variables

2^{nd} Order Runge-Kutta Methods

$$y_{n+1} = y_n + (a\boldsymbol{k_1} + b\boldsymbol{k_2})h$$

$$\boldsymbol{k_1} = f(x_n, y_n) \qquad \boldsymbol{k_2} = f(x_n + \alpha h, y_n + \beta \boldsymbol{k_1} h)$$

$$a + b = 1 \qquad \alpha b = \frac{1}{2} \qquad \beta b = \frac{1}{2}$$

Heun's Method

$$\boldsymbol{a} = \frac{\boldsymbol{1}}{\boldsymbol{2}}, b = \frac{1}{2}, \alpha = 1, \beta = 1$$

$$y_{n+1} = y_n + h\left(\frac{1}{2}\boldsymbol{k_1} + \frac{1}{2}\boldsymbol{k_2}\right)$$

$$\boldsymbol{k_1} = f(x_n, y_n)$$

$$\boldsymbol{k_2} = f(x_n + h, y_n + \boldsymbol{k_1} h)$$

Mid-Point Method

$$\boldsymbol{a} = \boldsymbol{0}, b = 1, \alpha = \frac{1}{2}, \beta = \frac{1}{2}$$

$$y_{n+1} = y_n + h\boldsymbol{k_2}$$

$$\boldsymbol{k_1} = f(x_n, y_n)$$

$$\boldsymbol{k_2} = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\boldsymbol{k_1} h\right)$$

Ralston's Method

$$\boldsymbol{a} = \frac{\boldsymbol{1}}{\boldsymbol{3}}, b = \frac{2}{3}, \alpha = \frac{3}{4}, \beta = \frac{3}{4}$$

$$y_{n+1} = y_n + h\left(\frac{1}{3}\boldsymbol{k_1} + \frac{1}{3}\boldsymbol{k_2}\right)$$

$$\boldsymbol{k_1} = f(x_n, y_n)$$

$$\boldsymbol{k_2} = f\left(x_n + \frac{3}{4}h, y_n + \frac{3}{4}\boldsymbol{k_1} h\right)$$

Example

Use 2nd Order Ralston Runge-Kutta Method to solve the following with $h = 1.5$ to get $y(3)$

$$\frac{dy}{dx} = y' = 3e^{-x} - 0.4y, \qquad y(0) = 5$$

$$y_{actual}(0.5) = -0.8196$$

$$y_{n+1} = y_n + h\left(\frac{1}{3}\boldsymbol{k_1} + \frac{1}{3}\boldsymbol{k_2}\right)$$

$$\boldsymbol{k_1} = f(x_n, y_n)$$

$$\boldsymbol{k_2} = f(x_n + \frac{3}{4}h, y_n + \frac{3}{4}\boldsymbol{k_1}h)$$

i	x_n	y_n	k_1	k_2	y_{n+1}
0	0.0	5	1.000	-1.476	4.024
1	1.5	4.024	-0.9402	-0.9692	2.5847
2	3.0	$\mathbf{2.5847}$			

4^{th} Order Runge-Kutta Methods

They are most widely used and are derived similar to the second-order ones. Greater complexity results from having to compare terms through h^4, and gives a set of 11 equations in 13 unknowns. The set of 11 equations can be solved with 2 unknowns being chosen arbitrarily.

$$y_{n+1} = y_n + \frac{h}{6}(\boldsymbol{k_1} + 2\boldsymbol{k_2} + 2\boldsymbol{k_3} + \boldsymbol{k_4})$$

$$\boldsymbol{k_1} = f(x_n, y_n)$$

$$\boldsymbol{k_2} = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\boldsymbol{k_1}h)$$

$$\boldsymbol{k_3} = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\boldsymbol{k_2}h)$$

$$\boldsymbol{k_4} = f(x_n + h, y_n + \boldsymbol{k_3}h)$$

Example

Use 4th Order RK method to solve the following with $h = 0.1$ to get $y(0.5)$

$$\frac{dy}{dx} = y' = -2x - y, \qquad y(0) = -1$$

$$y_{actual}(0.5) = -0.8196$$

$$y_{n+1} = y_n + \frac{h}{6}(\boldsymbol{k_1} + 2\boldsymbol{k_2} + 2\boldsymbol{k_3} + \boldsymbol{k_4})$$

$$\boldsymbol{k_1} = f(x_n, y_n)$$

$$\boldsymbol{k_2} = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\boldsymbol{k_1}h)$$

$$\boldsymbol{k_3} = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\boldsymbol{k_2}h)$$

$$\boldsymbol{k_4} = f(x_n + h, y_n + \boldsymbol{k_3}h)$$

x_n	y_n	k_1	k_2	k_3	k_4
0.0	-1.0000	1.000	0.850	0.858	0.714
0.1	-0.9145	0.715	0.579	0.586	0.456
0.2	-0.8562	0.456	0.333	0.340	0.222
0.3	-0.8225	0.222	0.111	0.117	0.011
0.4	-0.8110	0.011	-0.090	-0.085	-0.181
0.5	$\mathbf{-0.81959}$				

4th Order RK Method is **further gain in accuracy** with **less effort** than with the Taylor series method, and certainly is better than the Euler or modified Euler methods